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Asymptotic Study of a Problem of Axial Shear of a Cylindrical Tube

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Abstract

We study in this paper the problem at the limits of a cylindrical tube subjected to shear. The mathematical considerations of this problem lead to a non-linear differential equation. Resolution techniques allow us to have an analytical solution whose asymptotic stability we study. We then established that the different behaviours of the tube depend only on the constitutive law of the material through the coefficients, which themselves depend on the derivatives of the potential concerning the invariants.

Keywords - *Hyperelastic, compressibility, shear, nonlinearity, perturbation, asymptotic stability.*

I. INTRODUCTION

The problem of the telescopic shearing of a compressible hyperplastic cylinder has often been studied. Simple shear, pure shear, and telescopic shear are considered the most important modes in the deformation of materials [1]. While pure shear is an ideal deformation mode for metal forming operations [2], telescopic shear is considered as an optimal mode of deformation for grain refinement via severe plastic deformation. It is well known from the literature that, in the absence of body force, several deformations can be supported in equilibrium in an (in) compressible isotropic nonlinearly elastic solid material by applying surface traction alone [3]. Such deformations are said to be controllable. If within a given class of materials, the deformation is controllable for all materials and independent of any specific constitutive law in the considered class, then the deformation is said to be universal [4]. The deformation of particular interest in the present paper is a telescopic shear of a cylindrical circular tube.

Several authors have studied many deformations: finite extension, inflation, and torsion for isotropic materials from many different perspectives in the past.

For compressible isotropic materials, for which the finite extension, inflation, and torsion are not, in general, controllable. A class of materials admitting isochoric pure torsional deformation was proposed by several authors [5].

Few studies include the telescopic shear in the compressible case, and few authors have published an exact solution in this situation. However, there has been a resurgence of interest in determining solutions within the context of specific combined problems which are complicated to obtain, even in the isotropic case and for simple geometries. To solve these boundary problems, several authors use the perturbation method.

Perturbation methods, also known as asymptotic, allow the simplification of complex mathematical problems. The use of perturbation theory will allow approximate solutions to be determined for issues that cannot be solved by traditional analytical methods. Second-order ordinary linear differential equations are solved by engineers and scientists routinely. However, in many cases, real-life situations can require much more difficult mathematical models, such as non-linear differential equations.

Singularly perturbed differential equations, being an adequate mathematical model of real-life multi-time-scale systems, were studied extensively in the literature [6,7,8]. One of the essential classes of such equations is the class of equations with small-time delays of the order of a small positive parameter ε multiplying a part of the derivatives in the system. Brief surveys of results in this topic can be found [8,9,10]. One of the essential issues studied in the theory of differential equations is stability [6,11]. Two approaches to studying the strength of the trivial solution to linear constant-coefficients differential systems (without and with time delays) are most spread in the literature. The first (classical one) is based on the spectrum analysis of the system. The second (more recent one) is a Lyapunov-method-based one leading to sufficient conditions in terms of linear matrix inequalities. Traditional asymptotic stability has been studied either by Lyapunov's direct method or by Poincare's geometric method. The first attempt to unify the two procedures was carried out by La Sallby, combining information obtained from natural and straightforward Liapunov's functions with information about geometric properties obtained from the invariance principle of the limit set.

The purpose of the present paper is to examine, in dynamic, the axial shear problem of a hollow circular cylindrical tube. The inner surface of the tube is bonded to a rigid cylinder, and a uniformly distributed axial shear traction is applied to the outer surface of the tube with zero radial traction maintained at the same surface. The tube is assumed sufficiently long so that end effects are negligible.

Since the material is compressible, there is no radial deformation.

In the present analysis, we apply perturbation theory to approximate solutions to engineering problems that would otherwise be intractable through traditional analytical methods.

II. FORMULATION

We consider the axial shear of a hollow circular cylindrical tube. The body is composed of an incompressible isotropic hyperelastic material with strain-energy density per unit volume. Since the early work of Rivlin [12], the axial shear problem for general incompressible materials has received some attention. The problem for compressible materials has been more widely investigated [13].

For the compressible tube, with the inner surface bonded to a rigid cylinder and a uniformly distributed axial shear traction applied to the outer surface, the deformation is that of pure axial shear described by

$$r = R,\ \ \theta = \alpha_0 \Theta,\ \ z = \lambda Z + w(R,t), \tag{2.1}$$

Where (R,Θ,Z) and (r,θ,z) are respectively the reference and the deformed positions of material particle, $w(R,t)$ is the axial displacement which defines the telescopic shear,

$R_i \leq R \leq R_o$, $R_i$and $R_o$denote, respectively, the inner and outer radius of the cylinder.

We consider a compressible and opened tube defined by the angle Θ_0.the parameter$\alpha_0 = \pi/\Theta_0$ and λ is the twist angle per unit unloaded length.

From (2.1), in the cylindrical system, a routine calculation gives the physical components of the deformation gradient**F**,

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_0 & 0 \\ w'(R,t) & 0 & \lambda \end{pmatrix}, \tag{2.2}$$

And the physical components of the left Cauchy-Green tensor

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & w'(R,t) \\ 0 & \alpha_0^2 & 0 \\ w'(R,t) & 0 & \lambda^2 + w'(R,t)^2 \end{pmatrix}, \tag{2.3}$$

Where $w'(R,t) = \partial w(R,t)/\partial R$.

The strain energy density per unit undeformed volume for an elastic and isotropic material is given by [14]:

$$W = W(I_1, I_2, I_3) = \frac{\alpha}{2}(I_1 - 3) + \frac{\beta}{2}(I_1 - 3)^2 + \frac{\gamma}{2}(I_2 - 3) + \frac{\delta}{2}(I_3 - 1)\ , \tag{2.4}$$

with principal invariants

$$I_1 = 1 + \alpha_0^2 + \lambda^2 + w'(R,t)^2\ ,$$

$$I_2 = \alpha_0^2 + \lambda^2 + \alpha_0^2\lambda^2 + \left(\alpha_0^2 + \frac{\lambda^2}{2}\right) w'(R,t)^2,$$

$$I_3 = \alpha_0^2\lambda^2.$$

And $\alpha, \beta, \gamma, \delta$ are the constants.

The corresponding response equation for the Cauchy stress tensor**T** is:

$$\mathbf{T} = \frac{2}{\sqrt{I_3}}\left[(I_2 W_2 + I_3 W_3)\mathbf{1} + W_1 \mathbf{B} - I_3 W_2 \mathbf{B}^{-1}\right],$$

where **1** is the identity tensor and $W_i = \partial W / \partial I_i, (i = 1,2,3)$.

We obtain

$$\begin{cases} T_{rr} = \frac{2}{\lambda\alpha_0}\left[(I_2W_2 + I_3W_3) + W_1 - I_3W_2\left(1 + \frac{w'^2}{\lambda^2}\right)\right] \\ T_{\theta\theta} = \frac{2}{\lambda\alpha_0}\left[(I_2W_2 + I_3W_3) + \alpha_0^2 W_1 - I_3W_2 \frac{1}{\alpha_0^2}\right] \\ T_{zz} = \frac{2}{\lambda\alpha_0}\left[(I_2W_2 + I_3W_3) + W_1(\lambda^2 + w'^2) - I_3W_2\left(\frac{1}{\lambda^2}\right)\right] \end{cases},$$

$$\begin{cases} T_{r\theta} = T_{\theta z} = 0 \\ T_{rz} = \frac{2}{\lambda\alpha_0}\left[W_1 w' + \frac{1}{\lambda^2} I_3 W_2 w'\left(\frac{1}{\lambda}\right)\right]. \end{cases}$$

With the equations (2.1), (2.4), and (2.6), and in the absence of body forces, the equation of motion is given by:

$$\begin{cases} \frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \\ \frac{\partial T_{zr}}{\partial r} + \frac{T_{zr}}{r} = \rho\ddot{w} \end{cases} \tag{2.8}$$

where $\ddot{w} = \partial^2 w(R,t)/\partial t^2$ and $\rho = \rho_0/\lambda\alpha_0$ is the density in the deformed configuration.

Taking into account the form of (2-4), (2-5), and (2-7), we can rewrite the equations of motion (2-8) in the form:

$$\begin{cases} w''w + \frac{C_0}{R}w'^2 + \frac{C_1}{R} = 0 \\ w''w + C_2 w'' + \frac{C_3}{R}w'^3 + \frac{C_4}{R}w' = C_5\ddot{w} \end{cases}$$

by combining these two equations, we get:

$$w'' + \frac{K_0}{R}w'^3 + \frac{K_1}{R}w'^2 + \frac{K_2}{R}w' + K_3\ddot{w} = 0.$$

where K_i, (i=0-3) are constants.

III. APPLICATION OF PERTURBATION METHODS

The great majority of problems that we have to solve in Physics are not exactly solvable. Because of this and taking into account that we live in the epoch of computers, the last decades have seen a lot of progress in developing techniques leading to approximate solutions. The perturbation method is used for not precisely solvable Hamiltonian problems when the Hamiltonian differs slightly from an exactly solvable one. The difference between the two Hamiltonians is known as the perturbation Hamiltonian. All perturbation methods are based on the smallness of the latter concerning both Hamiltonians.

The equation (2.10) admits an obvious solution there $w_0 = 0$.
Linearizing aboutw_0, we plug

$$w = w_0 + \varepsilon H(R,t)$$

into the equation (2.10), to obtain

$$\varepsilon H'' + \varepsilon^3 \frac{K_0}{R} H'^3 + \varepsilon^2 \frac{K_1}{R} H'^2 + \varepsilon \frac{K_2}{R} H' + \varepsilon K_3 \ddot{H} = 0. \quad (3.1)$$

Where H plays the role of perturbation, ε "1 is a small parameter qualifying the magnitude of disturbance.
Suppose that for the compact$C_R = [R_i\ ,\ R_0]$, there exist two locally L_1functions [15]
$r(t)$ And $\eta(t)$: $\mathbf{R}^+ \rightarrow \mathbf{R}^+$such that (s.t)
$\forall (R,t) \in C_R \times \mathbf{R}^+$,$\|H(R,t)\| \leq r(t)$ and $\forall \delta > 0, \exists \mu_H(\delta) > 0\ s.t.$ if $E \subset [t, t+1]$ is a measurable set with measure$< \mu(\delta)$ then $\int r(\tau) d\tau \leq \delta,\ \tau \in E$.
As is shown in [16], these hypotheses guarantee the pre-compactness of the equation (3.1) in the restrict sense and the uniqueness of the solution for (3.1).
From (3.1), we assume that $H(R,t)$ has the following form [17]:

$$H(R,t) = f(R) + g(R) \cos(\widetilde{w} t)$$

Assuming ε "1, we can neglect the terms $\varepsilon^p, p \geq 2$. Equation (3.1) becomes.

$$f'' + \frac{K_2}{R} f' + \left[g'' + \frac{K_2}{R} g' - K_3 \widetilde{w}^2 g \right] \cos(\widetilde{w} t) = 0, \forall t \geq 0. \quad (3.2)$$

We get a system of decoupled equations:

$$\begin{cases} f'' + \frac{K_2}{R} f' = 0 \\ g'' + \frac{K_2}{R} g' - K_3 \widetilde{w}^2 g = 0 \end{cases} \quad (3.3)$$

We find the so-called Bessel equations encountered in many physical problems, particularly those with cylindrical symmetry. The solutions obtained (Frobenius method) are given a series.

The first equation of the system (3.3) gives

$$f(R) = -\frac{A_0}{K_2} R^{(1-K_2)} + A_1. \quad (3.4)$$

The second equation of the system (3.3) is a modified Bessel equation. The solution is given by:

$$g(R) = R^{\nu} \left[A_2 I_\nu \left(R \widetilde{w} \sqrt{K_3} \right) + A_3 K_\nu \left(R \widetilde{w} \sqrt{K_3} \right) \right],$$

Where $\nu = \frac{1-K_2}{2}$, $I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n \geq 0} \left(\frac{\left(\frac{x}{2}\right)^{2n}}{\Gamma(n+\upsilon+1)\, n!} \right)$,

$K_\nu(x) = \frac{\pi}{2 \sin(\nu\pi)} (I_{-\nu}(x) - I_\nu(x))$,$\nu \notin \mathbf{Z}$,

and A_i, $(i = 0,1,2,3)$ are constants.

A function is called a Bessel function (of fractional order) if it is a solution of the Bessel's differential equation (3.3.b), where ν is a positive no integral number. Bessel's differential equation plays a significant role in physics and engineering.
Thus, disturbed solution, expressing the deformation described in (2.1) is given by

$$\begin{aligned} W(R,t) &= \varepsilon . f(R) + \varepsilon . g(R) . \cos(\widetilde{w} . t) \\ &= \varepsilon \Big[-\frac{A_0}{K_2} R^{(1-K_2)} + A_1 + \\ &(R^\nu) \left[A_2 I_\nu \left(R \widetilde{w} \sqrt{K_3} \right) + A_3 K_\nu \left(R \widetilde{w} \sqrt{K_3} \right) \right]) \cos(\widetilde{w} . t) \Big], \end{aligned}$$

IV. ON STABILITY FOR PERTURBED DIFFERENTIAL EQUATIONS

Stability or instability of non-linear systems can often be tested by an approximate procedure, which leads to a linear equation describing the growth of the difference between the test solution and its neighbors.

We will now solve the approximate Bessel's differential equation in a class of analytic function, C_K.

Let $\nu = \frac{1-K_2}{2}$ be a positive no integral number and let p be a no negative integer with $p < \upsilon < p+1$.
Assume that a function $g \in C_K$ satisfies the differential inequality [21]

$$\left| g''(R) + \frac{K_2}{R} g'(R) - K_3 \widetilde{w}^2 g(R) \right| \leq \zeta,$$

for all $R \in C_R = [R_i\ ,\ R_0]$ And for some $\zeta \geq 0$. The sequence$b_n = \frac{1}{\Gamma(n+\upsilon+1)\, n!}$ satisfies the condition

$$b_{n+2} = O(b_n)\ \ as\ n\ \rightarrow\ \infty,$$

then there exists an approximate solution $g_h(R)$ of the Bessel's differential equation (3.3.b) such that

$$|g(R) - g_h(R)| \leq \zeta K L_\upsilon M(R)$$

where $L_\upsilon = \sum_{m \geq 0} \left(\frac{1}{(m-\upsilon)^2} \right) < \infty$,

$$M(R) \leq max\left\{\frac{R^{R+1}}{|\upsilon^2-n^2|^{R/2}};\frac{R^{R+1}}{|\upsilon^2-(n+1)^2|^{R/2}}\right\} \leq max\left\{\frac{R_o^{1+R_o}}{|\upsilon^2-n^2|^{R_i/2}};\frac{R_o^{1+R_o}}{|\upsilon^2-(n+1)^2|^{R_i/2}}\right\}.$$

The last term of this double inequality is constant and positive, and the hypotheses of the theorem on the global attractivity and the eventual stability are verified [19, 20].

Then, in virtue of this theorem, we obtain that the solution $g(R)$ is globally attractive and eventually stable with respect to (3.3b).

The stability of the cylindrical tube is investigated concerning superposed pure homogeneous deformations. Because of the simplicity of the underlying assumptions, necessary and sufficient conditions are obtained for stability in terms of the general strain-energy function for compressible, isotropic, homogenous, elastic materials.

In mathematics, any continuous function on a bounded closed interval is uniformly continuous and bounded.

The solution $w(R,t)$ is then bounded in $[R_i, R_o]$.

There is, therefore, a constant such L_0 that $|g(R) - g_h(R)| \leq L_0$.

The series defined in the solutions are convergent. On the other hand, they are uniformly continuous on the compact$[R_i\ ,\ R_0]$ They are bounded. We can then affirm that there is a constant$\widetilde{L_0}$, independent of R such that:

$$|w(R_1,t) - w(R_2,t)| \leq \widetilde{L_0}, \qquad \forall\, R_j \in [R_i, R_o], j = 1,2\ and\ \forall\ t \geq 0 \tag{3.9}$$

With Lipchitz's theorem, we can say that the solution defined in (3.6) is stable for the deformation (2.1).

The strain energy, described in (2.4), is polyconvex.

In general, under polyconvexity assumptions, no claim can be made as to the stability or smoothness of the solution, apart from the natural statement that the minimizer lies in the Sobolev space considered. Moreover, it is not known that minimizing deformation is a weak solution to the local balance equation due to possible singularities in the deformation gradient. We remark that polyconvexity implies the existence of all boundary conditions and body forces, which might be somewhat unrealistic.

V. CONCLUSION

We studied the behavior of a tube subjected to an axial shear problem. This study led us to the resolution of a non-linear partial differential equation. In this paper, we describe techniques for obtaining approximations to periodic time solutions of nearly second-order differential equations subject to a harmonic forcing term. The approximations take the form of an expansion in integer powers of a small parameter, having coefficients that are functions of time. We wish to be able to take a few terms of expansion and to be able to say that for some small fixed numerical value ofε supplied in a practical problem, the truncated series is close to the required solution for the whole range of the independent variable in the differential equation. The solution, in serial form, allowed us to study the stability of the tube.

In light of this study, we have shown the possibility offered by this method for the study of the stability of a hollow tube through a non-linear differential equation presenting an obvious solution. It emerges from this study that the stability analysis depends on the constitutive law of the material through the coefficients. The results obtained are significant because they suggest that it is possible to find a stability result by studying the character of Lipchitz's functions. This is in line with Cauchy - Lipchitz's theorem.

This approach is generalizable to any constitutive law of the same nature.

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